GCE: 503, Analysis and measure theory

January 2017

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Exercise 1:

Consider the sequence of functions f_n defined on the non-negative reals by $f_n(x) = 2nxP(x)e^{-nx^2}$, where P is a polynomial function.

(i). Is f_n pointwise convergent on $[0, \infty)$? Is f_n uniformly convergent on $[0, \infty)$? Explain your answers to both questions.

(ii). Let g_n be a sequence of continuous functions defined on $[0, \infty)$ and valued in $\mathbb{R}$. Assume that each g_n is in $L^1([0, \infty))$ and that the sequence g_n is uniformly convergent to zero. Prove or disprove: $\lim_{n\to\infty} \int_0^\infty g_n = 0$.

(iii). Determine (with proof) $\lim_{n\to\infty} \int_0^\infty f_n$.

Exercise 2 (all answers require proofs):

Let f_n be the sequence in $L^2(\mathbb{R})$ defined by $f_n = 1_{[n,n+1]}$.

(i). Let g be in $L^2(\mathbb{R})$. Does $\int f_n g$ have a limit as n tends to infinity?

(ii). Does the sequence f_n converge in $L^2(\mathbb{R})$?

Exercise 3:

Let X be a metric space. For any subset A of X, we denote by $\overline{A}$ the closure of A and $\mathring{A}$ the union of all open subsets contained in A. We set $\partial A = \overline{A} \setminus \mathring{A}$.

(i). Show that A is closed if and only if $\partial A \subset A$.

(ii). Show that A is open if and only if $\partial A \cap A = \emptyset$.

(iii). Is the identity $\partial(\partial B) = \partial B$ valid for all subsets B of X?

(iv). Show that if A is closed then $\partial(\partial A) = \partial A$.

Exercise 4:

Let X be a measure space, f_n a sequence in $L^1(X)$ and f an element of $L^1(X)$ such that f_n converges to f almost everywhere and $\lim_{n\to\infty} \int |f_n| = \int |f|$. Show that $\lim_{n\to\infty} \int |f_n - f| = 0$.